1. A chicken lays n eggs. Each egg independently does or doesn’t hatch, with probability p of hatching. For each egg that hatches, the chick does or doesn’t survive (independently of the other eggs), with probability s of survival. Let N ⇠ Bin(n, p) be the number of eggs which hatch, X be the number of chicks which survive, and Y be the number of chicks which hatch but don’t survive (so X + Y = N). Find the marginal PMF of X, and the joint PMF of X and Y . Are they independent?

Solutiin

To solve the problem, we will first define the relevant random variables and then derive the marginal PMF of \( X \) and the joint PMF of \( X \) and \( Y \).

### Definitions:

1. Let \( N \) be the number of eggs that hatch, where \( N \sim \text{Bin}(n, p) \). This means \( N \) follows a binomial distribution with parameters \( n \) (the total number of eggs) and \( p \) (the probability of hatching).

2. Each hatched chick survives with probability \( s \) and does not survive with probability \( 1 – s \).

3. Let \( X \) be the number of chicks that survive.

4. Let \( Y \) be the number of chicks that hatch but do not survive.

### Relationships:

From the above definitions, we have the relationships:

- \( X + Y = N \)

- \( Y = N – X \)

### Marginal PMF of \( X \):

To find the marginal PMF of \( X \), we note that given \( N = k \), the number of surviving chicks \( X \) follows a binomial distribution with parameters \( k \) and \( s \):

\[

X | N = k \sim \text{Bin}(k, s)

\]

Thus, the conditional PMF is given by:

\[

P(X = x | N = k) = \binom{k}{x} s^x (1 – s)^{k – x}

\]

The marginal PMF of \( X \) can be found using the law of total probability:

\[

P(X = x) = \sum\_{k = x}^{n} P(X = x | N = k) P(N = k)

\]

Where \( P(N = k) \) is given by the binomial PMF:

\[

P(N = k) = \binom{n}{k} p^k (1 – p)^{n – k}

\]

Thus,

\[

P(X = x) = \sum\_{k = x}^{n} \binom{k}{x} s^x (1 – s)^{k – x} \binom{n}{k} p^k (1 – p)^{n – k}

\]

This sum gives the marginal PMF of \( X \).

### Joint PMF of \( X \) and \( Y \):

To find the joint PMF \( P(X = x, Y = y) \), we recognize that since \( Y = N – X \):

\[

P(X = x, Y = y) = P(X = x, N = x + y) = P(X = x | N = x + y) P(N = x + y)

\]

Given \( N = k \), the conditional distribution is still:

\[

P(X = x | N = k) = \binom{k}{x} s^x (1 – s)^{k – x}

\]

Thus, we have:

\[

P(X = x, Y = y) = P(X = x | N = x + y) P(N = x + y)

\]

This can be expressed as:

\[

P(X = x, Y = y) = \binom{x+y}{x} s^x (1 – s)^{y} \cdot \binom{n}{x+y} p^{x+y} (1 – p)^{n – (x+y)}

\]

### Independence of \( X \) and \( Y \):

To check if \( X \) and \( Y \) are independent, we need to see if:

\[

P(X = x, Y = y) = P(X = x) P(Y = y)

\]

Since \( Y \) is directly related to \( X \) by the relationship \( Y = N – X \), \( X \) and \( Y \) are not independent. The values of \( X \) directly determine the values of \( Y \).

### Conclusion:

* The marginal PMF of \( X \) is given by:

\[

P(X = x) = \sum\_{k = x}^{n} \binom{k}{x} s^x (1 – s)^{k – x} \binom{n}{k} p^k (1 – p)^{n – k}

\]

* The joint PMF of \( X \) and \( Y \) is given by:

\[

P(X = x, Y = y) = \binom{x+y}{x} s^x (1 – s)^{y} \cdot \binom{n}{x+y} p^{x+y} (1 – p)^{n – (x+y)}

\]

* \( X \) and \( Y \) are not independent.